Decomposition theory of the *U*(1) gauge potential and the London assumption in topological quantum mechanics

2002 J. Phys.: Condens. Matter 14 7941

(http://iopscience.iop.org/0953-8984/14/34/312)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **14** (2002) 7941–7947 PII: S0953-8984(02)37981-5

Decomposition theory of the $U(1)$ gauge potential and the London assumption in topological quantum mechanics

Yi-shi Duan, Xin Liu[1] **and Peng-ming Zhang**

Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, People's Republic of China

E-mail: liuxin@lzu.edu.cn

Received 11 June 2002
Published 15 August 2002
Online at stacks.iop.org/JPhysCM/14/7941

Abstract
The decomposition theory of the $U(1)$ gauge potential has been studied. A rigorous proof of the London assumption is given. By making use of this gauge potential decomposition and the ϕ-mapping topological current theory, the precise expression for $\vec{\nabla} \times \vec{V}$ is obtained, and the topology in quantum mechanics is discussed.

1. Introduction

In recent years, the decomposition theory of the gauge potential has been playing a more and more important role in both theoretical physics and mathematics, and has become an important aspect of topological field theory [1, 2]. One of the authors (Duan) has been engaged in study in this field for a long time, and has made much progress in many problems using this decomposition theory, such as the decomposition of the $SO(N)$ spin connection, the structure of the GBC topological current, the decomposition of the $SU(N)$ connection and the effective theory of $SU(N)$ QCD [3–7].

In this paper the decomposition theory of the $U(1)$ electromagnetic gauge potential in terms of the condensate wavefunction has been studied. This decomposition reveals the inner structure of the gauge potential, and establishes a direct relationship between differential geometry and the topology of the gauge field. Based on this decomposition, a direct relationship between the electromagnetic gauge potential and the velocity field in quantum mechanics is given, which is just the London assumption in superconductivity [8]. Furthermore, by making use of this gauge potential decomposition and the ϕ-mapping topological current theory, the precise topological expression for $\vec{\nabla} \times \vec{V}$ is obtained, and the vortices in quantum mechanics are characterized by the ϕ-mapping topological numbers—Hopf indices and the Brouwer degrees.

[1] Author to whom any correspondence should be addressed.

2. Decomposition of the $U(1)$ gauge potential

In the theory of superconductivity, the condensate wavefunction $\psi(x)$ is the order parameter of the charged continuum, which is a section of the complex line bundle. The $U(1)$ covariant derivative $D_i\psi$ and its complex conjugate $(D_i\psi)^*$ are introduced to describe the interaction between ψ and the electromagnetic field:

$$D_i\psi = \partial_i\psi - \mathrm{i}\frac{e}{\hbar c}a_i\psi, \tag{1}$$

$$(D_i\psi)^* = \partial_i\psi^* + \mathrm{i}\frac{e}{\hbar c}a_i\psi^*, \tag{2}$$

where $i = 1, 2, 3$ and a_i is the magnetic gauge potential vector. The magnetic field tensor is given by

$$f_{ij} = \partial_i a_j - \partial_j a_i. \tag{3}$$

Multiplying (1) with ψ^* and (2) with ψ, it is easy to find the decomposition expression for the $U(1)$ gauge potential:

$$a_i = \frac{\hbar c}{2\mathrm{i}e}\frac{1}{\psi^*\psi}(\psi^*\partial_i\psi - \partial_i\psi^*\psi) - \frac{\hbar c}{2\mathrm{i}e}\frac{1}{\psi^*\psi}(\psi^* D_i\psi - (D_i\psi)^*\psi). \tag{4}$$

To study the meaning of the second term in the RHS of (4), we write $\psi(x)$ in terms of two real functions $\phi^1(x)$ and $\phi^2(x)$:

$$\psi(x) = \phi^1(x) + \mathrm{i}\phi^2(x), \tag{5}$$

and define a two-dimensional unit vector

$$n^a = \frac{\phi^a}{\|\phi\|} \qquad (a = 1, 2) \tag{6}$$

where $\|\phi\|^2 = \phi^a\phi^a = \psi^*\psi$; then (4) can be expressed as

$$a_i = \frac{\hbar c}{e}\epsilon_{ab}n^a\partial_i n^b - \frac{\hbar c}{e}\epsilon_{ab}n^a D_i n^b. \tag{7}$$

Let

$$k^a = \epsilon^{ab}n^b \tag{8}$$

be another two-dimensional unit vector which is orthogonal to n^a:

$$k^a n^a = 0, \qquad k^a k^a = 1; \tag{9}$$

then using n^a and k^a, equation (4) is rewritten as

$$a_i = -\frac{\hbar c}{e}(k^a\partial_i n^a - k^a D_i n^a). \tag{10}$$

Let u^a be a unit vector field satisfying

$$D_i u^a = 0 \qquad (u^a u^a = 1) \tag{11}$$

and expressed as

$$u^a = \cos\theta n^a + \sin\theta k^a; \tag{12}$$

it can be proved that

$$-k^a D_i n^a = \partial_i\theta. \tag{13}$$

Therefore the covariant derivative part of (10) is identified as the gradient of a phase factor θ, and (4) can be re-expressed as

$$a_i = \frac{\hbar c}{2\mathrm{i}e}\left[\frac{1}{\psi^*\psi}(\psi^*\partial_i\psi - \partial_i\psi^*\psi) + \partial_i\theta\right]. \tag{14}$$

We see that the second term of (14), $\frac{\hbar c}{2ie}\partial_i\theta$, behaves as a $U(1)$ gauge transformation of a_i, which contributes nothing to the gauge field tensor f_{ij} defined by (3) and can be ignored in $U(1)$ decomposition theory [4, 9, 10]. Therefore the decomposition of the $U(1)$ gauge potential is simplified as

$$a_i(\psi) = \frac{\hbar c}{2ie}\frac{1}{\psi^*\psi}(\psi^*\partial_i\psi - \partial_i\psi^*\psi), \tag{15}$$

where $a_i(\psi)$ means that the magnetic gauge potential a_i possesses an inner structure in terms of charged condensate wavefunction ψ and ψ^*. Formula (15) is a fundamental expression in $U(1)$ topological quantum mechanics.

3. The London assumption

It is well known that, as semi-phenomenological scenarios of the low-dimensional BEC continuum, the non-linear Gross–Pitaevskii (GP) equation for superfluid and the Ginzburg–Landau (GL) equation for superconductivity are important. The GP equation is known to be [11]

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\partial_i^2\psi + U(x)\psi + \frac{4\pi\hbar^2 a}{m}|\psi|^2\psi, \tag{16}$$

and the current is defined as

$$J_i = \rho V_i, \tag{17}$$

where ρ is the density, $\rho = |\psi|^2$. The GL equations for superconductors are given by [12]

$$\frac{1}{2m}\left(-i\hbar\partial_i - \frac{e}{c}A^i_{ext}\right)^2\psi + a\psi + b|\psi|^2\psi = 0, \tag{18}$$

$$(\vec{\nabla}\times\vec{B})^i = \frac{4\pi}{c}J_i, \tag{19}$$

where the current J_i is covariant under $U(1)$ gauge transformation:

$$J_i = e\rho V_i - \frac{e^2}{mc}\rho A^i_{ext}, \tag{20}$$

with A^i_{ext} the external electromagnetic potential.

In both (17) and (20), the velocity field is defined as in fundamental quantum mechanics:

$$V_i = \frac{\hbar}{2im}\frac{1}{\psi^*\psi}(\psi^*\partial_i\psi - \partial_i\psi^*\psi). \tag{21}$$

Comparing (15) with (21) we find an important inner relationship between the velocity field V_i and the electromagnetic gauge potential $a_i(\psi)$:

$$a_i(\psi) = \frac{mc}{e}V_i; \tag{22}$$

this is just the London assumption [8]. This gives the mechanism whereby the creation of the intrinsic electromagnetic field is due to the motion of condensate wavefunction ψ; i.e., the magnetic field

$$\vec{B} = \vec{\nabla}\times\vec{a} \tag{23}$$

is produced by $\vec{\nabla}\times\vec{V}$. This is the essence and significance of the London assumption. Finally, it must be pointed out that the above proof of the London assumption necessarily depends on the decomposition theory of the $U(1)$ gauge potential.

4. The topology in quantum mechanics

In traditional quantum mechanics, the wavefunction is usually expressed in the form

$$\psi = |\psi| e^{i\Theta(\vec{x})}, \tag{24}$$

and the velocity field $\vec{V}$ becomes the gradient of the phase factor $\Theta(\vec{x})$:

$$\vec{V} = \frac{\hbar}{m}\vec{\nabla}\Theta, \tag{25}$$

which directly leads to a trivial curl-free result:

$$\vec{\nabla} \times \vec{V} = 0. \tag{26}$$

By the London assumption (22), this means that the curl motion of the charged wavefunction can never produce a magnetic field. Nearly half a century ago, Onsager and Feynman found that the statement $\vec{\nabla} \times \vec{V} = 0$ must be modified, and Landau predicted that $\vec{\nabla} \times \vec{V}$ can be non-zero at a singular vortex line, and behaves as a δ-function [13]. Therefore, it is indispensable to study what the exact expression for $\vec{\nabla} \times \vec{V}$ is in topological field theory.

In this paper, using the unit vector n^a defined in (6), the expression for the velocity field (21) is rewritten as

$$V^i = \frac{\hbar}{m}\epsilon_{ab} n^a \partial_i n^b, \tag{27}$$

and we can find a non-zero expression for $\vec{\nabla} \times \vec{V}$:

$$(\vec{\nabla} \times \vec{V})^i = \frac{\hbar}{m}\frac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j n^a \partial_k n^b. \tag{28}$$

Noticing that $\partial_i n^a = \frac{\partial_i \phi^a}{\|\phi\|} + \phi^a \partial_i \frac{1}{\|\phi\|}$, and using the Green function relation in ϕ-space

$$\Delta_\phi \ln \|\phi\| = 2\pi \delta^2(\vec{\phi}), \qquad \left(\Delta_\phi = \frac{\partial}{\partial \phi^a}\frac{\partial}{\partial \phi^a}\right), \tag{29}$$

it can be proved that [3, 5]:

$$(\vec{\nabla} \times \vec{V})^i = \frac{h}{m}\delta^2(\vec{\phi}) D^i\left(\frac{\phi}{x}\right), \tag{30}$$

where $D^i(\frac{\phi}{x}) = \frac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j \phi^a \partial_k \phi^b$ is the Jacobian vector. The above formula with the singular function $\delta^2(\vec{\phi})$ is just the precise topological expression for $\vec{\nabla} \times \vec{V}$ that Landau and Feynman expected to find. This expression provides an important conclusion:

$$\vec{\nabla} \times \vec{V} \begin{cases} = 0, & \text{iff } \vec{\phi} \neq 0; \\ \neq 0, & \text{iff } \vec{\phi} = 0. \end{cases} \tag{31}$$

According to the ϕ-mapping topological current theory [14], the two-dimensional topological current is defined as

$$j^i = \frac{1}{2\pi}\frac{1}{2}\epsilon^{ijk}\epsilon_{ab}\partial_j n^a \partial_k n^b = \delta^2(\vec{\phi}) D^i\left(\frac{\phi}{x}\right); \tag{32}$$

thus $\vec{\nabla} \times \vec{V}$ can be expressed as a topological current:

$$(\vec{\nabla} \times \vec{V})^i = \frac{h}{m} j^i. \tag{33}$$

The implicit function theory shows that [15], under the regular condition $D^i(\phi/x) \neq 0$, the general solutions of

$$\phi^1(x, y, z) = 0, \qquad \phi^2(x, y, z) = 0 \tag{34}$$

can be expressed as

$$x = x_j(s), \qquad y = y_j(s), \qquad z = z_j(s), \tag{35}$$

which represent the N isolated singular strings $L_j (j = 1, 2, \ldots, N)$ with parameter s. In quantum mechanics these singular strings are just the vortex lines. In δ-function theory [16], one can prove

$$\delta^2(\vec{\phi}) = \sum_{j=1}^{N} \beta_j \int_{L_j} \frac{\delta^3(\vec{x} - \vec{x}_j(s))}{|D(\frac{\phi}{u})|_{\Sigma_j}} \, \mathrm{d}s, \qquad (i = 1, 2, 3) \tag{36}$$

where $D(\frac{\phi}{u})_{\Sigma_j} = (\frac{1}{2}\epsilon^{jk}\epsilon_{mn}\frac{\partial\phi^m}{\partial u^j}\frac{\partial\phi^n}{\partial u^k})$, and Σ_j is the jth planar element transverse to L_j with local coordinates (u^1, u^2). The positive integer β_j is the Hopf index of ϕ-mapping. Meanwhile, the direction vector of L_j is given by

$$\left.\frac{\mathrm{d}x^i}{\mathrm{d}s}\right|_{x_j} = \left.\frac{D^i(\phi/x)}{D(\phi/u)}\right|_{x_j}. \tag{37}$$

Then from (36) and (37) we find the important inner topological structure of $\vec{\nabla} \times \vec{V}$:

$$(\vec{\nabla} \times \vec{V})^i = \frac{h}{m} j^i = \frac{h}{m} \sum_{j=1}^{N} \beta_j \eta_j \int_{L_j} \frac{\mathrm{d}x^i}{\mathrm{d}s} \delta^3(\vec{x} - \vec{x}_j(s)) \, \mathrm{d}s, \tag{38}$$

where η_j is the Brouwer degree, $\eta_j = \operatorname{sgn} D(\phi/u) = \pm 1$. From (38), the winding number of $\vec{\phi}$ for around L_j is

$$W_j = \beta_j \eta_j; \tag{39}$$

therefore the vorticity of the vortex line L_j is

$$\Gamma_j = \int_{\Sigma_j} \vec{\nabla} \times \vec{V} \cdot \mathrm{d}\vec{s} = \frac{h}{m} W_j, \tag{40}$$

and the total vorticity on a surface Σ should be

$$\Gamma = \int_{\Sigma} \vec{\nabla} \times \vec{V} \cdot \mathrm{d}\vec{s} = \frac{h}{m} \sum_{j=1}^{N} W_j. \tag{41}$$

This is the topological essence of the vorticity in quantum mechanics.

In the superconductivity theory, using the unit vector n^a, the expression for the magnetic gauge potential vector (15) can be rewritten as

$$a_i = \frac{\hbar c}{e} \epsilon_{ab} n^a \partial_i n^b, \tag{42}$$

and the gauge field tensor f_{ij} is

$$f_{ij} = \frac{\hbar c}{e} \epsilon_{ab} \partial_i n^a \partial_j n^b.$$

f_{ij} can be re-expressed in a δ-function form:

$$f_{ij} = \frac{\hbar c}{e} 2\pi \epsilon_{ijk} \delta^2(\vec{\phi}) D^k\left(\frac{\phi}{x}\right) = 2\pi \frac{\hbar c}{e} \epsilon_{ijk} j^k, \tag{43}$$

where j^i is the two-dimensional topological current defined in (32). The intrinsic magnetic field defined from f_{ij} is given by

$$B^i = \tfrac{1}{2}\epsilon^{ijk} f_{jk} = \Phi_0 j^i, \tag{44}$$

where $\Phi_0 = \frac{hc}{e}$ is the unit flux quantum; therefore,

$$B^i = \Phi_0 \delta^2(\vec{\phi}) D^i\left(\frac{\phi}{x}\right). \tag{45}$$

From (38) we find that there are N singular vortex lines contributing to B^i:

$$B^i = \Phi_0 \sum_{j=1}^{N} W_j \int_{L_j} \frac{\mathrm{d}x^i}{\mathrm{d}s} \delta^3(\vec{x} - \vec{x}_j(s))\,\mathrm{d}s, \tag{46}$$

where each vortex line L_j carries a magnetic flux $\Phi_j = W_j \Phi_0$, and this leads to the phenomenon of magnetic flux quantization:

$$\Phi = \int_\Sigma B^i\,\mathrm{d}\sigma_i = \Phi_0 \sum_{j=1}^{N} W_j. \tag{47}$$

This is the topological essence of flux quantization.

For the GL theory, there is a relation [12, 17]

$$\vec{B} - \lambda^2 \nabla^2 \vec{B} = \frac{mc}{e} \vec{\nabla} \times \vec{V}, \tag{48}$$

where λ is the penetration depth, $\lambda^2 = \frac{mc^2}{4\pi\rho e^2}$; in the London approximation, λ is a constant. From (38) we have

$$\vec{B} - \lambda^2 \nabla^2 \vec{B} = \Phi_0 \sum_{j=1}^{N} W_j \int_{L_j} \frac{\mathrm{d}\vec{x}}{\mathrm{d}s} \delta^3(\vec{x} - \vec{x}_j(s))\,\mathrm{d}s. \tag{49}$$

We see that in the simple case $W_j = 1$, equation (49) is just the so-called modified London equation [17, 18]. This means that when the condensate wavefunction ψ has no zero values, $\vec{\phi} \neq 0$, i.e., $\delta^2(\vec{\phi}) = 0$, and $\vec{B} - \lambda^2 \nabla^2 \vec{B} = 0$, which just corresponds to the Meissner state; while in the case of a mixed state, $\vec{\phi}$ possesses N isolated zeros, $\delta^2(\vec{\phi}) \neq 0$, and thus a type-II superconductor is penetrated by an array of N vortices, which are created from the zero points of $\vec{\phi}$ and carry a quantum flux proportional to the winding number W_j.

Acknowledgments

This work was supported by the National Natural Science Foundation and the Doctor Education Fund of the Educational Department of the People's Republic of China.

References

[1] Faddeev L and Niemi A J 1999 *Phys. Rev. Lett.* **82** 1624
Faddeev L and Niemi A J 1999 *Phys. Lett.* B **449** 214
Faddeev L and Niemi A J 1999 *Phys. Lett.* B **464** 90
[2] Cho Y M, Lee H and Pak D G 1999 *Preprint* hep-th/9905215
Cho Y M 1999 *Preprint* hep-th/9906198
[3] Duan Y S and Ge M L 1979 *Sci. Sin.* **11** 1072
Duan Y S and Meng X H 1993 *J. Math. Phys.* **34** 1149
[4] Duan Y S, Zhang H and Li S 1998 *Phys. Rev.* B **58** 125
Duan Y S and Zhang H 1999 *Eur. Phys. J.* D **5** 47
[5] Duan Y S, Li S and Yang G H 1998 *Nucl. Phys.* B **514** 705
[6] Duan Y S and Fu L B 1998 *J. Math. Phys.* **39** 4343
[7] Li S, Zhang Y and Zhu Z Y 2000 *Phys. Lett.* B **487** 201
[8] Feynman R P, Leighton R B and Sands M 1965 *Lectures on Physics* vol 3 (Reading, MA: Addison-Wesley)
[9] Fu L B, Duan Y S and Zhang H 2000 *Phys. Rev.* D **61** 045004

[10] Duan Y S and Zhang S L 1992 *Int. J. Eng. Sci.* **30** 153
Duan Y S, Yang G H and Jiang Y 1997 *Gen. Rel. Grav.* **29** 715
[11] Gross E P 1961 *Nuovo Cimento* **20** 454
Pitaevskii L P 1961 *JETP* **13** 451
[12] Lifshitz E M and Pitaevskii L P 1980 *Statistical Physics* vol 2 (Oxford: Pergamon)
Schrieffer J R 1964 *Theory of Superconductivity* (New York: Benjamin-Cummings)
[13] Onsager L 1949 *Nuovo Cimento* **6** 249
Feynman R P 1955 *Progress in Low Temperature Physics* vol 1, ed C G Gorter (Amsterdam: North-Holland) pp 17–51
[14] Duan Y S, Fu L B and Jia G 2000 *J. Math. Phys.* **41** 4379
[15] Goursat É 1904 *A Course in Mathematical Analysis* vol 1 (transl. E R Hedrick) (New York: Dover)
[16] Schouton J A 1951 *Tensor Analysis for Physicists* (Oxford: Clarendon)
[17] de Gennes P G 1966 *Superconductivity of Metals and Alloys* (New York: Benjamin)
Brandt E G 1995 *Rep. Prog. Phys.* **58** 1465
[18] Duan Y S, Zhang H and Li S 1998 *Phys. Rev.* B **58** 125
Duan Y S, Zhang H and Li S 1999 *Eur. Phys. J.* D **5** 47
Zhang H, Duan Y S and Hu B B 2000 *Europhys. Lett.* **52** 101